$\overline{MS}$ Perturbation Theory and the Higgs Boson Mass

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Abstract

We show that $\overline{MS}$ perturbation theory develops tachyonic singularities for some value of the dimensional regularization scale μ unless the physical Higgs mass exceeds some (cutoff dependent) minimum value.

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For the history of the subject we refer to reviews [1] and quote only recent papers containing the latest refinements of the conventional approach [2],[3]. Once the experimental lower bound on the top-quark mass exceeded $80\,GeV$, attention shifted from the original Linde-Weinberg bound, based on the properties of the one-loop effective potential for small ϕ, close to the minimum, to the large ϕ behavior of the effective potential , as determined by renormalization group (RG) considerations.

$$V_{eff}(\phi) = \frac{1}{4}\overline{\lambda}(t)(\xi(t)\phi)^4$$

Here, $\xi(t)$ is the anomalous dimension factor, and $\overline{\lambda}(t)$ is the $\bar{MS}$ running coupling constant .

$$t = \ln\frac{\phi}{M_0}, \qquad \frac{d\overline{\lambda}(t)}{dt} = \beta_\lambda(\overline{\lambda}(t), \overline{g}(t))$$

It is then argued that vacuum stability requires $\bar{\lambda}\ (t) > 0$ up to some high scale, $\phi \sim M_{GUT}, \mathrm{or} M_{Pl}, (M_0 \sim M_Z, \mathrm{or}\ m_t,$ or $246 GeV$).To implement this condition, one has to know (or approximate) the $\beta-$ function, integrate the RG differential equations starting from some initial values,$\bar{\lambda}\ (0), \bar{g}^2(0)$, and relate the smallest acceptable $\bar{\lambda}\ (0)$ to a physical Higgs mass. (In the latest effective potential calculation [3] the condition is $\overline{\lambda}_{eff}(t) > 0$ where $\bar{\lambda}_{eff}$ differs perturbatively from $\bar{\lambda}$).

We propose a new approach. It is also of the perturbative RG variety, but is not based on the effective potential . The results are found to be insensitive to scale ambiguity. The essential input is that one is perturbing about the correct vacuum. A neccessary condition for this is that the vev of the (shifted) field be zero, order by order in perturbation theory, and the $\bar{MS}$ renormalized mass squared in the $\bar{MS}$ propagator of the shifted field be positive.

We start by computing the relation between the perturbative pole mass and the $\bar{MS}$ mass, for both the Higgs boson and the t-quark. The relation follows from the perturbative definition of the pole mass,

$$0 = Re\,\overline{D}^{-1}(M^{*^2}) = M^{*^2} - \overline{M}^2 - Re\,\overline{\Sigma}(M^{*^2}) \tag{1}$$

In this equation, $\bar{D}(q^2)$, and $\bar{\Sigma}(q^2)$ are the two- point Green Function and self-energy function, renormalized according to the $\bar{MS}$ prescription. M^* is the perturbative pole

mass.The result is [5]

$$M^{*^2} = \overline{M}^2\{1+\overline{\lambda}(3\overline{I}_{00}(M^{*^2})+9\overline{I}_{\overline{MM}}(M^{*^2}))+N_c\overline{y}^2(\frac{M^{*^2}}{\overline{M}^2}-4\frac{\overline{m}^2}{\overline{M}^2})\overline{I}_{\overline{mm}}(M^{*^2})+2(\overline{\zeta}_v-1)\} \quad (2)$$

m is the t-quark mass, and y is the t-quark Yukawa coupling ($m = \frac{yv}{\sqrt{2}}$). The contributions from the electroweak gauge sector, proportional to g_2, g_1, have also been calculated, but are not written out here. they will be included below. The term $\overline{\zeta}_v - 1$ comes from a finite shift of the vev required in the $\bar{M}S$ scheme to enforce $<\hat{H}>= 0$ through one- loop order. It will cancel out of the ratio computed below, so we do not have to give its value here. [5]$\bar{I}_{ab}$ is the dimensionally regularized $\bar{M}S$ scalar one-loop two-point integral.

$$\begin{aligned}\overline{I}_{ab}(q^2) &= [\mu^{4-d} i \int \tfrac{d^d l}{(2\pi)^d} \tfrac{1}{(l^2-a^2)((l-q)^2-b^2)}]_{\overline{MS}} \\ &= \tfrac{1}{16\pi^2}[\ln\tfrac{ab}{\mu^2} + \int_0^1 dx \ln\tfrac{a^2x+b^2(1-x)-q^2x(1-x)}{ab}]\end{aligned} \quad (3)$$

Then (2) is

$$M^{*^2} = \overline{M}^2\{1+\frac{\overline{\lambda}}{16\pi^2}[12\ln\frac{M^2}{\mu^2}-24+3\sqrt{3}\pi]+N_c\frac{\overline{y}^2}{16\pi^2}(1-\frac{4}{r^2})[\ln\frac{m^2}{\mu^2}+f(r)]+2(\overline{\zeta}_v-1)\} \quad (4)$$

where

$$r = \frac{M}{m}, \qquad f(r) = -2+2\sqrt{\frac{4-r^2}{r^2}}\arctan\sqrt{\frac{r^2}{4-r^2}}$$

The corresponding calculation for the t-quark gives [4]

$$m^{*^2} = \overline{m}^2\{1+\frac{\overline{y}^2}{16\pi^2}[\frac{3}{2}\ln\frac{m^2}{\mu^2}+\Delta(r)]+\frac{\overline{g}_s^2}{16\pi^2}C_F(8-6\ln\frac{m^2}{\mu^2})+2(\overline{\zeta}_v-1)\} \quad (5)$$

where

$$\Delta(r) = -4+\frac{r^2}{2}+(\frac{3}{2}r^2-\frac{1}{4}r^4)\ln r^2+\frac{r}{2}(4-r^2)^{\frac{3}{2}}\arctan\sqrt{\frac{4-r^2}{r^2}}$$

We take the ratio of (2) to (5) and expand to one-loop order.

$$\begin{aligned}\tfrac{M^{*^2}}{m^{*^2}} = \tfrac{\overline{M}^2}{\overline{m}^2}\{1+\tfrac{\overline{\lambda}}{16\pi^2}[12\ln\tfrac{M^2}{\mu^2}-24+3\sqrt{3}\pi]+\tfrac{\overline{y}^2}{16\pi^2}[N_c(1-\tfrac{4}{r^2})(\ln\tfrac{m^2}{\mu^2}+f(r))-\tfrac{3}{2}\ln\tfrac{m^2}{\mu^2}-\Delta(r)] \\ +\tfrac{\overline{g}_s^2}{16\pi^2}C_F(6\ln\tfrac{m^2}{\mu^2}-8)+g_2^2, g_1^2\, terms + 2-loop\}\end{aligned} \quad (6)$$

The $\overline{\zeta}_v - 1$ terms, which also contain explicit dependence on $\ln\mu^2$, have cancelled out. A necessary condition for the $\bar{M}S$ perturbation calculations to be defined in the broken

symmetry phase is that $\bar{M}^2, \bar{m}^2$ be positive. Since the ratio of pole masses is positive, (6) satisfies the requirement perturbatively, for μ around the weak scale. For large μ^2, one has to provide a RG treatment of the large logarithms, just as in the conventional calculation involving the effective potential . In the broken symmetry phase, one can define the renormalized coupling constants such that the relation

$$\frac{M^2}{m^2} = 4\frac{\lambda}{y^2} \tag{7}$$

is exact when all the quantities are either "star" (on-shell renormalization scheme) or "bar"($\bar{M}S$ renormalization scheme)[4],[5]. Thus, not all quantities in (6) can be varied independently as functions of μ. we use (7) to eliminate $\bar{\lambda}$ appearing in (6). To leading (one-loop) order, the scale dependence of the ratio of $\bar{M}S$ masses is determined by the coefficients of the explicit $\ln\mu^2$ terms in (6). For the other masses in (6), the difference between "star" and "bar" is higher order (combined with explicitly two-loop effects),as is the implicit μ dependence of the "bar" coupling constants. After these observations, and reinstating the g_2^2, g_1^2 terms, differentiating (6), we obtain

$$\begin{aligned} \mu\tfrac{d}{d\mu}(\bar{\rho}) = \tfrac{\bar{y}^2}{16\pi^2}[6\bar{\rho}^2 + (2N_c - 3 + 12C_F\tfrac{\bar{g}_s^2}{\bar{y}^2})\bar{\rho} - 8N_c \\ -(\tfrac{9}{2}\tfrac{\bar{g}_2^2}{\bar{y}^2} + \tfrac{1}{6}\tfrac{\bar{g}_1^2}{\bar{y}^2})\bar{\rho} + 3\tfrac{\bar{g}_2^4}{\bar{y}^4} + \tfrac{3}{2}(\tfrac{\bar{g}_2^2+\bar{g}_1^2}{\bar{y}^2})^2] + 2 - loop \end{aligned} \tag{8}$$

where $\bar{\rho} = r^2 = \frac{\bar{M}^2}{\bar{m}^2}$.

Let the right hand side of (8) be denoted β_ρ. Because of the $-8N_c$ term in (8), there is a critical value of ρ below which β_ρ becomes negative. And if the starting value of ρ is below this value, as ρ decreases the derivative becomes more negative, driving ρ negatuve for some value of μ, unless some higher order effect intervenes.

The first higher order effect is the running of the $\bar{M}S$ coupling constants, which appear as coefficients in (8), and the dependence of the lower bound on the cutoff (maximum value of $\frac{\mu}{\mu_0}$). One has to integrate the coupled RG equations for five independent "coupling constants", $\bar{g}_s^2, \bar{g}_2^2, \bar{g}_1^2, \bar{y}^2, \bar{\rho}$. Let $t = \ln\frac{\mu}{\mu_0}$.

$$
\begin{aligned}
\frac{d}{dt}\overline{g}_s^2 &= -\frac{1}{16\pi^2}(22-\tfrac{4}{3}N_f)\overline{g}_s^4 \\
\frac{d}{dt}\overline{g}_2^2 &= -\frac{1}{16\pi^2}[\tfrac{44}{3}-\tfrac{4}{3}N_f-\tfrac{1}{3}N_d]\overline{g}_2^4 \\
\frac{d}{dt}\overline{g}_1^2 &= \frac{1}{16\pi^2}[\tfrac{20}{9}N_f+\tfrac{1}{3}N_d]\overline{g}_1^4 \\
\frac{d}{dt}\overline{y}^2 &= \frac{1}{16\pi^2}[(3+2N_c)\overline{y}^4-12C_F\overline{g}_s^2\overline{y}^2-\tfrac{9}{2}\overline{g}_2^2\overline{y}^2-\tfrac{17}{6}\overline{g}_1^2\overline{y}^2] \\
\frac{d}{dt}(\rho) &= \frac{\overline{y}^2}{16\pi^2}[6\rho^2+(2N_c-3+12C_F\tfrac{\overline{g}_s^2}{\overline{y}^2})\rho-8N_c \\
&\quad -(\tfrac{9}{2}\tfrac{\overline{g}_2^2}{\overline{y}^2}+\tfrac{1}{6}\tfrac{\overline{g}_1^2}{\overline{y}^2})\rho+3\tfrac{\overline{g}_2^4}{\overline{y}^4}+\tfrac{3}{2}(\tfrac{\overline{g}_2^2+\overline{g}_1^2}{\overline{y}^2})^2]
\end{aligned}
\tag{9}
$$

The first three equations are integrated trivially. If we neglect the $\bar{g}_2, \bar{g}_1$ contributions to the $\bar{y}$ running, that equation can also be integrated analytically. But if one runs up to high scales, the electroweak gauge couplings become of same order as the QCD coupling constant; so we use NDSolve from Mathematica to provide an interpolating function solution for $\bar{y}^2$ which is substituted into the $\bar{\rho}$ equation, which is again integrated numerically by NDSolve.

Before giving any results, we discuss the question of their sensitvity to the choice of dimensional regularization scale μ. In this approach, the scale sensitvity comes from the choice of the Electroweak scale μ_0. This enters in three ways: (i) the starting values of the $\bar{g}_i^2, \bar{y}^2$ at $t=0$ ($\mu=\mu_0$). (ii) the connection between t_{max} and the nominal value of the cutoff, $\mu_{max}=\Lambda$. (iii) the conversion of the found critical $\bar{\rho}_c(0)$ back to the ratio of pole masses, (6). It is clear that there will be substantial cancellation between these.

We take $m_t = 175$ GeV. For the low scale cutoff, we take $\Lambda = 1$ TeV. For a first orientation, we keep only the large coupling constants , g_s, y, in equations (9), ($g_2, g_1 \to 0$). To check the sensitivity to the choice of starting Electroweak scale μ_0, we solve the remaining three equations from (9) starting from $\mu_0 = M_Z$ ($t_{max} = 2.4$), and again, starting from $\mu_0 = m_t$ ($t_{max} = 1.74$). We also require as input the initial values $\bar{g}_s^2, \bar{y}^2$. For $mu_0 = M_Z$, We take $\bar{g}_s^2 = 1.483$ ($\alpha_s(M_Z) = .118$). For the Yukawa coupling constant we have

$$
v^*(M_W) = (\sqrt{2}G_F)^{-\frac{1}{2}}\,(1-\Delta r^*)^{-\frac{1}{2}} = 251\text{GeV}
$$

$$y^{*^2}(M_Z) = 2\frac{m^{*^2}}{v^{*^2}} = .972$$

Then we use equation (59) of [5] to convert from star to bar, giving the initial value $\bar{y}^2(M_Z) = .970$. With these initial values, the critical initial value of $\rho(0)$ for which $\rho(t)$ falls through zero at $t = 2.4$ is $\rho_c(0) = .26685 = (\bar{M}/\bar{m})_c^2$. Converting back to a ratio of pole masses by (6) gives the critical value $M_c^* = 74$ GeV.

For initial scale $\mu_0 = m_t$, we need $\bar{g}_s^2(m_t a)$ and $\bar{y}^2(m_t)$. For the QCD coupling constant , to our level of accuracy, we can simply run $\bar{g}_s^2$ from $\mu = M_Z$ to $\mu = m_t$ using the first equation of (9). This gives $\bar{g}_s^2(m_t) = 1.355 \quad (\alpha_s(m_t) = .1078)$. The determination of the Yukawa coupling constant is more complicated. We can proceed in two different orders. First, we can just run $\bar{y}^2(M_Z)$ to $\bar{y}^2(m_t)$ using the fourth equation of (9) and obtain $\bar{y}^2(m_t) = .905$. Alternatively, we can run $v^{*^2}(M_Z)$ to $v^{*^2}(m_t)$ (see equation (40) of ([5])) and convert to $y^{*^2}(m_t)$ and then to $\bar{y}^2(m_t)$ which gives $\bar{y}^2(m_t) = .877$. So we carry through the calculation for three different initial values: $\bar{y}^2(m_t) = .905, .891(avg), .877$. The resulting M_C^* are (resp) $74.7, 74.1, 73.5$. These results give us some confidence that the calculation is not sensitive to the choice of initial weak scale in the range from M_Z to m_t, as long as the scale dependence of the input parameters is handled consistently.

Having checked this point, we reinstate the gauge coupling constants in(9). We integrate this set of equations, starting at $\mu_0 = m_t$, with the additional input $g_2^2(m_t) = .4239$, $g_1^2(m_t) = .1260$. Then the critical initial value of $\rho(0)$ for which $\rho(t)$ falls through zero at $t = 1.74$ is .182. Converting back to a ratio of pole masses by equation (6) gives $M_c^* = 72$ GeV. We take this value as our best estimate of the smallest Higgs mass for which the $\overline{MS}$ perturbation theory is nontachyonic up to scale μ equal to one TeV (for $m_t = 175$ GeV).

Taking the large scale to be the Planck scale ($\mu_{max} \simeq 10^{19}, t_{max} \simeq 39$) and using the same input at $\mu_0 = m_t$, we find the critical value of the Higgs mass to be 140 GeV.

In conclusion, we find that the $\overline{MS}$ perturbation theory develops tachyonic singularities (negative mass squared in the $\overline{MS}$ renormalized propagator) when the dimensional regularization scale factor μ exceeds some assigned cutoff value unless the physical Higgs

mass exceeds some minimum value, which depends on the cutoff value.

The requirement that $\overline{MS}$ perturbation theory not develop tachyonic singularites below some prescribed cutoff scale is in principle different from and independent of the requirement that the effective potential not develop a stable minimum at some value of the vev much greater than the weak scale. However, until the recent papers of Casas,Espinosa, and Quiros [3] it has been generally taken that the requirement on the effective potential was practically equiivalent to the requirement that the $\overline{MS}$ running quartic coupling constant stay positve below the cutoff scale. And in the minimal Standard Electroweak Model, in conventional renormalization schemes, the condition of positive ratio of squared masses implies, and is implied by, the condition of positive quartic and squared Yukawa coupling constants (7). In this case the numerical results are not different (up to differences in handling input parameters,etc); and our numerical results are the same as those of Altarelli and Isadori [2]. But CEQ have argued that consideration of the scale dependence involved in the minimization of the perturbative effective potential leads to the requirement that$\overline{\lambda}_{eff}$ stay positive, where $\overline{\lambda}_{eff}$ is not the same as $\overline{\lambda}$. Then the numerical equivalence is broken and the results of the two approaches are different. It is also possible that the singular behavior we have found is just a pathology of $\overline{MS}$ perturbation theory, which is interesting in itself, but not justifcation for a restriction on the physical Higgs mass.

It is clearly desirable to have a large scale lattice simulation study of the combined Higgs-heavy quark-QCD sector. (Contributions from light quarks and electroweak gauge bosons are small, particularly if one doesn't run up to some very high scale). We note that a quenched approximation simulation is not adequate for this problem. The term in (8) which triggers the possible instability is the $-8N_c$, clearly a contribution from an internal fermion closed loop.

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